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The motion of a cylindrical body in a stratified fluid under the action of a radiation force

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ABSTRACT

The free motion of a thin cylindrical body is investigated based on a previously derived expression for the radiation force acting on moving point sources in a stratified fluid. The fundamental equations of motion are derived, the limits of applicability of the approximation used are indicated and the results of calculations of typical trajectories of a body which begins to move with a specified velocity from a position of neutral buoyancy at an angle to the horizon are presented. Calculations of the trajectory of motion of a thin cylindrical body in a stratified fluid when the total radiation force is taken into account show that the effect of the lateral component of this force is considerable and leads not only to quantitative corrections but also to qualitative effects (for example, to an increase in the oscillations of the body and a change in its direction of motion). The results obtained pertain both to the motion of solids in fluids and to the translational motion of vortex dipoles in weakly stratified media.

The motion of material objects in complex media possessing a spectrum of natural oscillations, generally speaking, is accompanied by the radiation of waves of one kind or another, which leads to the occurrence of additional radiation forces, acting on the sources of these waves. Effects of this kind have been investigated in detail in electrodynamics in the case of Cherenkov radiation of electromagnetic waves by rapidly moving charges. Similar physical phenomena also occur in stratified media, for example, in the atmosphere or in the ocean, when internal gravitational waves of moving bodies are radiated. However, the radiation of internal waves has a number of specific features due to their anisotropic nature,[1–3] connected with the action of the force of gravity.

For uniform motion of a source at an arbitrary angle to the horizon, due to asymmetry of the radiation field with respect to the direction of motion, the total radiation force acting on the source, in addition to the longitudinal component (the wave drag) also has a transverse component, which does not do work, but bends the trajectory of motion of the source. Both these components of the radiation force were calculated for a specified source velocity in Ref. 4. It is of interest to consider the self-consistent problem of the accelerated or retarded motion of bodies in a stratified medium when they are acted upon by the total radiation force in addition to the force of gravity.

The purpose of this paper is to investigate the trajectories of motion of a body in such a medium in the quasi-stationary approximation.

1. Fundamental equations

General relations. We will consider the motion in an ideal incompressible stratified fluid of a body of cylindrical shape of radius a, the axis of which is perpendicular to the density gradient. For simplicity, we will assume the fluid to be unbounded and exponentially stratified in density, so that the Brunt-Vaisala frequency is constant:

$$N \equiv \left(-\frac{g}{\rho_f}\frac{d\rho_f}{dz}\right)^{1/2} = \text{const}$$

where g is the acceleration due to gravity and $\rho_f(z)$ is the unperturbed fluid density. Suppose that, at the initial instant of time, the body has a velocity v_0, directed at an angle φ_0 to the horizon. We will investigate the subsequent motion of the body.

We will choose a fixed system of coordinates x_s and z_s so that the level $z_s = 0$ is the level of neutral buoyancy of the body, i.e. at this level $\rho_f(0) = \rho_s$, where ρ_s is the body density. The fluid density is then given by $\rho_f(z_s) = \rho_s \exp(-N^2 z_s/g)$.

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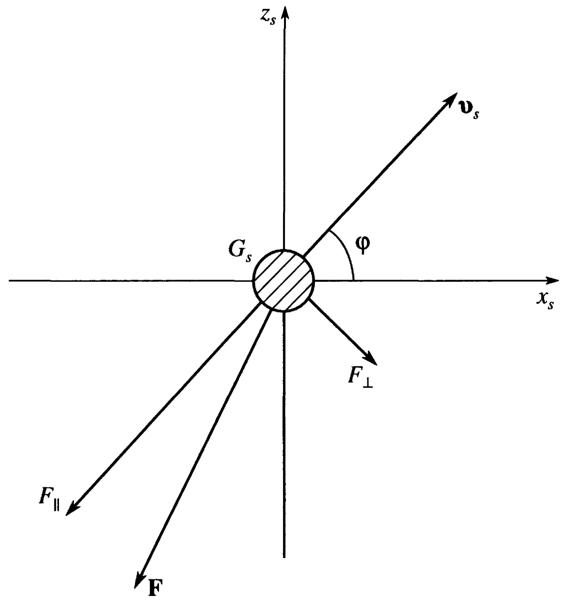

Fig. 1.

The equation of motion of the body has the form

$$\rho_s \frac{d\mathbf{v}_s}{dt} = \int_S (p\mathbf{n}, d\mathbf{s}) + \rho_s \mathbf{g} G_s \tag{1.1}$$

where v_t is the body velocity, G_s is the area of cross section of the body, p is the pressure of the surrounding fluid, S is the body surface and $\mathbf{n}$ is the unit vector of the normal to the body surface.

The first term on the right-hand side of Eq. (1.1) is the force acting on the body from the fluid. To calculate it we will change to a system of coordinates moving with the body (Fig. 1). In this non-inertial system of coordinates the fluid at considerable distances from the body moves with velocity $-v_s(t)$, and Euler's equation takes the form

$$\rho_f\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v}\right] + \nabla p = -\rho_f \frac{d\mathbf{v}_s}{dt} \tag{1.2}$$

We will represent the fluid velocity field in the form of the sum of the velocities of the unperturbed flow and of a perturbation introduced by the body into the flow:

$$\mathbf{v} = -\mathbf{v}_s + \mathbf{v}'; \quad -\mathbf{v}_s = (U, W), \quad \mathbf{v}' = (u', w')$$

We will model the immersed body by a mass dipole, oriented along the free stream, with a moment $I \sim \upsilon_s G_s$, where $G_s \sim a^2$. In particular, if the moving body is a cylinder of radius a, the value of the dipole moment will be $l = 2\pi a^2 \upsilon_s$. Such a model, as is well known,[2] is justified if the Froude number is sufficiently high: $Fr \equiv \upsilon_s/(Na) \gg 1$.

To find the components of the radiation force, which arises due to the loss of momentum by the body to the radiation of internal waves, it is necessary to know the velocity perturbation field. Assuming that the amplitude of the waves radiated by the body is fairly small in the far zone ($r \gg \lambda$, where λ is the wavelength), we will use the linear approximation to describe them. In the system of coordinates considered, the equations of motion and continuity, describing these perturbations in the Boussinesq approximation, have the form

$$\frac{Du'}{Dt} + \frac{1}{\rho_{fs}}\frac{\partial p'}{\partial x} = 0, \quad \frac{Dw'}{Dt} + \frac{1}{\rho_{fs}}\frac{\partial p'}{\partial z} + \frac{\rho' g}{\rho_{fs}} = 0, \quad \frac{D\rho'}{Dt} + w'\frac{d\rho_f}{dz} = 0$$
$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = m(x, z); \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + U(t)\frac{\partial}{\partial x} + W(t)\frac{\partial}{\partial z} \tag{1.3}$$

Here u' and w' are the perturbations of the horizontal and vertical components of the velocity, p′ and ρ' are the pressure and density perturbations, ρ_{fs} is the fluid density at the point where the body is situated, $m(x,z)$ is a mass source, that models the motion of the body, which, in the case of a moving dipole considered, is equal to $l\delta'(r_\parallel)\delta(r_\perp)$, where δ is the Dirac delta function, δ' is its derivative, and $r_\parallel$ and $r_\perp$ are the longitudinal and transverse components of the radius vector of the source with respect to its velocity vector. In Fig. 1 we show the direction of motion of the source and the components of the radiation force $F_\parallel$ and $F_\perp$ acting on it, and also the total radiation force $\mathbf{F}$. It is obvious, in view of the linearity of Eqs (1.3), that all the perturbations of the hydrodynamic fields are proportional to l.

Using a Fourier transformation of the horizontal and vertical components, we can reduce system (1.3) to a single equation for the complex amplitude of the perturbation of the vertical component of the velocity $\tilde{w}'(t, k_x, k_z)$:

$$\frac{D_F^2\tilde{w}'}{D_Ft^2} + \frac{k_x^2}{k_x^2+k_z^2}N^2\tilde{w}' = -i\frac{k_z}{k_x^2+k_z^2}\frac{D_F^2\tilde{m}(k_x,k_z)}{D_Ft^2} \tag{1.4}$$

where $D_F/D_Ft = \partial/\partial t + ik_xU + ik_zW$ is the operator of the substantional derivative D/Dt in the Fourier representation of the spatial coordinates, $\tilde{m}(k_x, k_z)$ is the Fourier transform of the source, and k_x and k_y are the components of the wave vector $\mathbf{k}$.

It can be seen from Eq. (1.4) that when the condition

$$\left|\frac{\partial\tilde{w}'}{\partial t}\right| \ll |(k_xU + k_zW)\tilde{w}'| \tag{1.5}$$

is satisfied, its solution has the form

$$\tilde{w}' \simeq \frac{-ik_z\tilde{m}(k_x,k_z)}{k_x^2+k_z^2-\left[\dfrac{k_xN}{k_xU(t)+k_zW(t)}+i\varepsilon\right]^2} \tag{1.6}$$

where ε is an infinitesimal quantity, introduced in accordance with Lighthill's rule[2] to eliminate the singularity in the solution. It is obvious that, when condition (1.5) is satisfied, the time-dependence of the wave field becomes purely parametric. The limits of applicability of this quasi-steady approximation will be estimated below.

To calculate the force acting on the body we will use the method described previously in Refs 5 and 4. The hydrodynamic force acting on unit length of a cylindrical body in a fluid is determined by the flux of momentum removed to infinity by internal waves from an isolated volume G, containing the body, by the change in momentum in unit time inside this volume, and also by the momentum of the external forces – of inertia and gravitation, acting on the fluid in this volume. We will introduce a numbering of the coordinates so that the subscript 1 corresponds to the x coordinate and the subscript 2 corresponds to the z coordinate. We can then write the following expression for the j-th component ($j = 1, 2$) of this force

$$f_j = -\frac{\partial}{\partial t}\iint\limits_G \rho_f\upsilon_j' ds + \int\limits_{ABCD}\int(\Pi_{j1}n_1 + \Pi_{j2}n_2)ds - \iint\limits_G\rho_f\left(\frac{\partial\upsilon_{sj}}{\partial t} + g\delta_{j2}\right)ds \tag{1.7}$$

where $\Pi_{j,l}(j,l = 1, 2)$ are the elements of the momentum flux tensor, n_l are the components of the unit vector of the normal to the surface, bounding the isolated cylindrical volume G, ABCD is a rectangular contour enveloping the section of this cylindrical volume in the perpendicular plane x_s, z_s, and δ_{j2} is the Kronecker delta.

If the body is replaced by a mass source $m(x,z)$, we have from Euler's equation, taking into account the equation of continuity for each j-th velocity component,

$$\frac{\partial\upsilon_j'}{\partial t} + \left(\frac{\partial\Pi_{j1}}{\partial x_1} + \frac{\partial\Pi_{j2}}{\partial x_2}\right) = -\rho_f\frac{\partial\upsilon_{sj}}{\partial t} - g\rho_f\delta_{j2} + \rho_f\upsilon_j' m \tag{1.8}$$

Integrating Eq. (1.8) over the cross-section area inside the isolated fluid volume G, with the exception of the cross-section area G_s of the body itself, and equating the result to expression (1.7), we obtain, in the Boussinesq approximation,

$$\mathbf{f} = \rho_{fs}\int\limits_{G_s}\frac{\partial\boldsymbol{\upsilon}'}{\partial t}ds - \rho_{fs}\mathbf{g}G_s - \rho_{fs}\iint\limits_{-\infty}^{+\infty}\boldsymbol{\upsilon}' m\,dx\,dz$$

The first term on the right-hand side of this equality is the force related to the added mass of the body in unsteady fluid motion.[1,3] It is equal to $-c_m\rho_{fs}\partial\upsilon_s/\partial t$, where c_m is the added mass coefficient. The second term is the Archimedes buoyancy force and the third term is the required radiation force due to the radiation of waves by the body.

A detailed calculation of the radiation force acting on uniformly moving sources in a stratified medium was carried out previously in Ref. 4. Here the dependences of the total radiation force and its components on the value and direction of the velocity of the source are given by the formulae

$$\mathbf{F} = -\frac{\rho_{fs}l^2N^3}{4\pi\upsilon_s^3}\mathbf{R}(\varphi) \tag{1.9}$$

The components of the vector $\mathbf{R}$ are

$$R_x = -\int\limits_{-\pi/2}^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{\cos(\theta-\varphi)}d\theta = \frac{4}{3}\cos\varphi - 2\Phi(\varphi)\cos^2\varphi - 2\cos^3\varphi + \Psi(\varphi)\cos^4\varphi$$

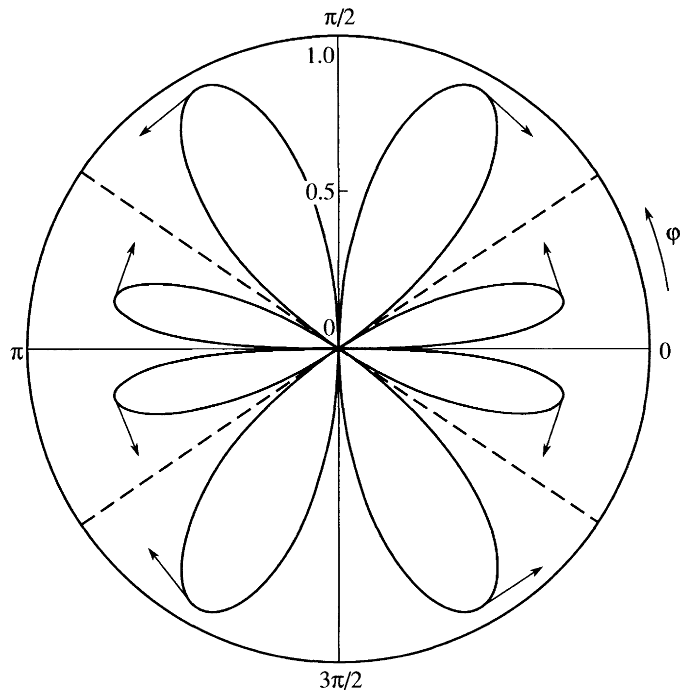

Fig. 2.

$$R_z = -\int_{-\pi/2}^{\pi/2} \frac{\sin^3\theta\cos\theta}{\cos(\theta-\varphi)}d\theta = -2\sin\varphi\left[\frac{1}{3} + \cos^2\varphi - \Phi(\varphi)\cos^3\varphi\right]$$

$$\Phi(\varphi) = \ln\left|\frac{1+\sqrt{2}\cos(\varphi+\pi/4)}{1-\sqrt{2}\cos(\varphi+\pi/4)}\right|, \quad \Psi(\varphi) = \ln\left|\frac{1-\cos\varphi}{1+\cos\varphi}\right|$$

The integrals are understood in the sense of the principal value, while the integration is carried out over all possible directions of the wave vector $\boldsymbol{k} = (k_x, k_z)$.

Hence we obtain for the components of the vector $\boldsymbol{R}$

$$R_{\parallel} = -\frac{2}{3}, \quad R_{\perp}(\varphi) = -\sin 2\varphi\left(1 + \frac{1}{2}\Psi(\varphi)\cos\varphi\right) \tag{1.10}$$

Note that previously in Ref. 4 only a quadrature formula was derived for $R_{\perp}(\varphi)$ and the corresponding dependence was constructed numerically. Here we have succeeded in obtaining an analytic expression for it. As can be seen, the value of the wave drag, equal to the longitudinal component of the radiation force, $F_{\parallel} \sim R_{\parallel}$, is independent of the direction of motion, whereas the transverse component of the radiation force $F_{\perp} \sim R_{\perp}$ depends on it. The angular dependence of the transverse component of the radiation force, normalized to the wave drag, is shown in Fig. 2 in polar coordinates. The arrows indicate the direction of action of the lateral force in the corresponding angular sectors. The local extrema in the first quadrant occur when $\varphi \approx 0.217$ and $\varphi \approx 1.078$. It is interesting to note that, in addition to the obvious directions $\varphi = 0$, π and $\varphi = \pi/2$, in which the lateral component of the radiation force vanishes due to the symmetry of the radiation field, there is one other direction in each quadrant $\varphi \approx 0.585 \pm n\pi/2$ ($n = 0, \ldots, 3$), in which this force also vanishes. These directions are indicated by the dashed lines in Fig. 2. It is in these directions (rather than in the horizontal or vertical directions) that, for small deviations, a restoring force, directed towards it, occurs (see the arrows in Fig. 2).

Formulae (1.9) and (1.10) remain true when there is a fairly slow change in the velocity of the body motion, i.e., in the quasi-steady approximation, the limits of applicability of which are estimated below.

2. The equations of motion of a cylinder taking the radiation force into account

We will now write the closed self-consistent system of equations of the free motion of a cylindrical body (a mass source) in a uniformly stratified medium (N = const) when acted upon by a radiation force

$$\frac{dx_s}{dt} = \upsilon_{sx}, \quad \frac{dz_s}{dt} = \upsilon_{sz}$$

$$(m_s + m_a)\frac{d\upsilon_{sx}}{dt} = F_x(\boldsymbol{\upsilon}_s), \quad (m_s + m_a)\frac{d\upsilon_{sz}}{dt} = F_z(\boldsymbol{\upsilon}_s) + (\rho_{fs} - \rho_s)gG_s \tag{2.1}$$

Here $m_s = \rho_s G_s$ is the mass of the body and $m_a = \rho_{fs} G_s$ is the added mass of the body[1] (the added mass coefficient of the cylinder is equal to unity).

We will introduce the following dimensionless variables

$$\xi = x_s/a, \quad \zeta = z_s/a, \quad \tau = Nt, \quad \mathbf{u} = \mathbf{v}_s/(Na) \tag{2.2}$$

and we will write the fundamental system of equations for the modulus of the body velocity u and the angle φ, defining the direction of motion of the body (Fig. 1):

$$\frac{d\xi}{d\tau} = u\cos\varphi, \quad \frac{d\zeta}{d\tau} = u\sin\varphi$$
$$\frac{du}{d\tau} = \frac{-1}{1+e^{-\zeta/b}}\left[\frac{2e^{-\zeta/b}}{3u} + b(1-e^{-\zeta/b})\sin\varphi\right]$$
$$\frac{d\varphi}{d\tau} = \frac{-1}{u(1+e^{-\zeta/b})}\left[\frac{-R_\perp(\varphi)e^{-\zeta/b}}{u} + b(1-e^{-\zeta/b})\cos\varphi\right] \tag{2.3}$$

where $b = g/(N^2 a)$ is the dimensionless scale of variation of the unperturbed density of the medium, while the relation $R_\perp(\varphi)$ is expressed by formula (1.10).

It is also necessary to supplement this system of equations with the initial conditions

$$\xi(0) = \xi_0, \quad \zeta(0) = \zeta_0, \quad u(0) = u_0, \quad \varphi(0) = \varphi_0$$

where the initial position of the body on the x axis, defined by the dimensionless coordinate ξ_0, is unimportant in view of the uniformity of the problem along the x axis, and we can therefore henceforth put $\xi_0 = 0$, which we will do. The equation for the horizontal coordinate ξ is then split, so that, by obtaining u and φ by solving the residual equations of system (2.3), we can then determine ξ.

The governing equations in this form only contain one dimensionless parameter b, the role of which in fact is also unimportant since, as will be seen later, we will use the approach developed here when this parameter is considerably greater than unity. This enables us to take the limit as $b\to\infty$ in system (2.3), after which no parameters, apart from the initial data, remain in it. The fundamental system of equations then finally reduces to the following three equations

$$\frac{d\zeta}{d\tau} = u\sin\varphi, \quad \frac{du}{d\tau} = -\frac{1}{3u} - \frac{\zeta}{2}\sin\varphi, \quad \frac{d\varphi}{d\tau} = \frac{R_\perp(\varphi)}{2u^2} - \frac{\zeta}{2u}\cos\varphi \tag{2.4}$$

In this system there is a first integral

$$u^2 + \frac{\zeta^2}{2} = -\frac{2}{3}\tau + E_0; \quad E_0 = u_0^2 + \frac{\zeta_0^2}{2} \tag{2.5}$$

where E_0 is the initial value of the total energy of the body. It can be seen from formula (2.5) that the total energy decreases linearly with time due to radiation losses and vanishes after a finite time $T_0 = 3E_0/2$. Naturally, only the longitudinal component of the radiation force facilitates the energy losses, whereas the transverse component does no work.

One more useful equation for the components of the kinetic energy, related to the motion along the x axis, follows from system (2.4):

$$\frac{d(u\cos\varphi)^2}{d\tau} = G(\varphi) \equiv -\cos^2\varphi\left[\frac{2}{3} + R_\perp(\varphi)\operatorname{tg}\varphi\right] \tag{2.6}$$

Hence we see that the loss of kinetic momentum along the x axis is due both to the longitudinal and transverse components of the radiation force. Moreover, an analysis of this formula shows that, due to the transverse component of the radiation force, the function $G(\varphi)$ becomes positive (i.e., the momentum of the body along the x axis increases) for angles $\varphi > 0.89$. A graph of $G(\varphi)$ is shown in Fig. 3 (curve 1), where, for comparison, we show the same relation when $R_\perp(\varphi)$, when the lateral force is ignored (curve 2).

3. The area of applicability of the above approach

We will estimate the limits of applicability of the system of equations employed. The quasi-steady approximation holds when the characteristic time scale N^{-1} of the variation of the velocity of the body, oscillating under the action of the force of gravity and the Archimedes force, is large compared with the time a/v_s taken for the wave field to become established in a frame of reference connected with the body (we can convince ourselves of this by formally substituting expression (1.6) into condition (1.5)). This condition agrees with the condition of applicability of replacing the body by a dipole source; in both cases the body velocity must be sufficiently high or, more accurately, the Froude number must be high.

On the other hand, the Boussinesq approximation used earlier holds when the characteristic scale of variation of the density g/N^2 is much greater than the characteristic wavelength v_s/N radiated by the body, which in dimensionless variables, gives $u \ll b$.

Both these limitations reduce to the double inequality

$$1 \ll u \ll b \tag{3.1}$$

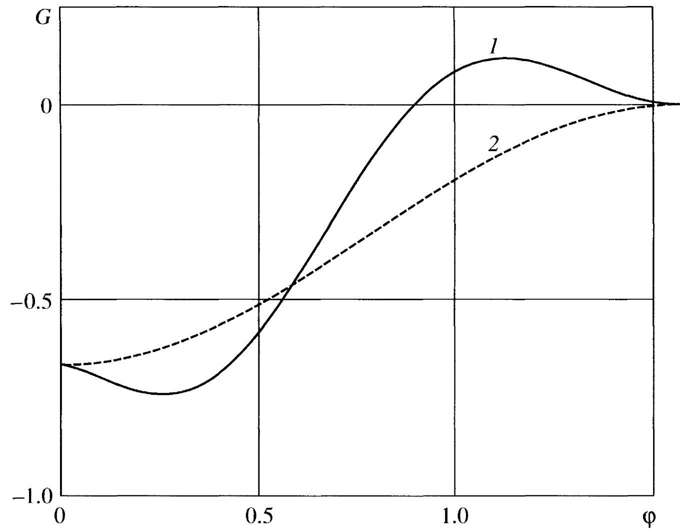

Fig. 3.

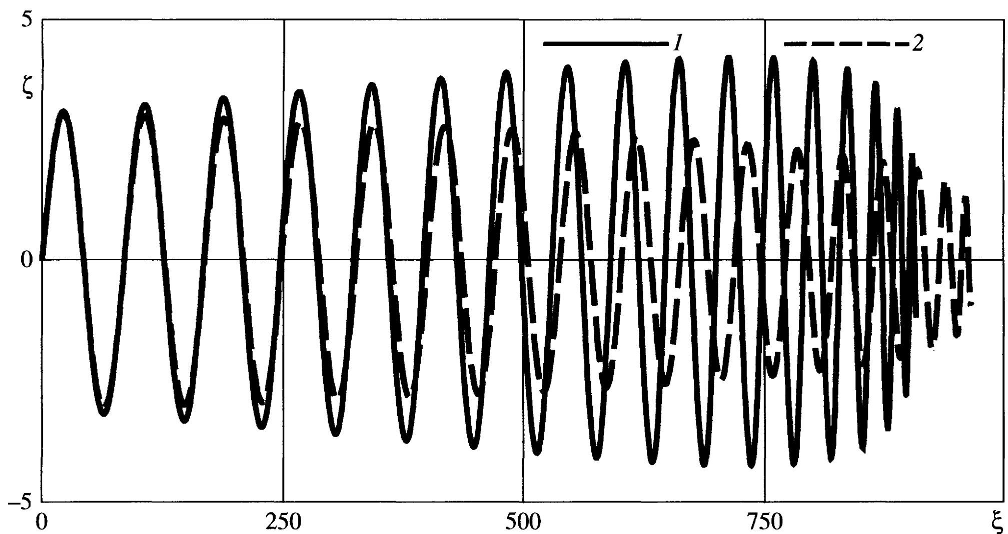

Fig. 4.

We recall that the dimensionless parameter $b = g/(N^2 a)$ is equal to the ratio of the scale of the density variation to the body radius, which, in all cases of practical importance, is in fact extremely high (for example, the characteristic scale of the density variation in the atmosphere is 8 km), so that, for the dimensionless velocity u (or for the Froude number) there is a fairly wide range of admissible values.

4. Calculation of the trajectories of motion of a cylinder

Since system of Eq. (2.4) cannot be solved analytically, its typical solutions were analysed by numerical integration. We will initially consider the motion of the body with a specified initial velocity $u_0 = 10$ from a position of neutral buoyancy $\zeta = 0$ at an angle $\varphi_0 = 0.217$ to the x axis (here the transverse component of the radiation force reaches a local maximum and is directed away from the x axis). In Fig. 4 we show the trajectories of the body motion both taking the action of the transverse component of the radiation force into account (curve 1), and without it (curve 2), but taking into account the longitudinal force of the wave drag. (Note that, when the radiation force is ignored, the motion of the body consists of non-attenuating oscillations, the amplitude and period of which is approximately the same as on the initial part of the trajectory.) It can be seen that, under the influence of the lateral force, the amplitude of the oscillations of the body initially increases considerably, and then falls sharply to zero, whereas the action of only one longitudinal component of the radiation force leads to a monotonic attenuation of the oscillations. This effect is more noticeable the smaller the initial angle of inclination of the velocity vector to the horizontal axis.

Moreover, when the total radiation force is taken into account during the attenuation time T_0, the body travels along a shorter path along the horizontal than when only one wave drag force acts. The reduction in the distance travelled along the horizontal agrees with formula (2.6), since, for small angles φ, the function $R_\perp(\varphi)$ is positive (see also the behaviour of the function $G(\varphi)$ in Fig. 3), i.e., the attenuation of the x component of the momentum is more pronounced.

In Fig. 5 we show the time dependences of the vertical coordinate φ of the body, the velocity modulus u and the angle φ, which defines the direction of the body motion. In agreement with formula (2.5) the oscillations are cut off when $\tau = T_0 = 150$, when the total energy of

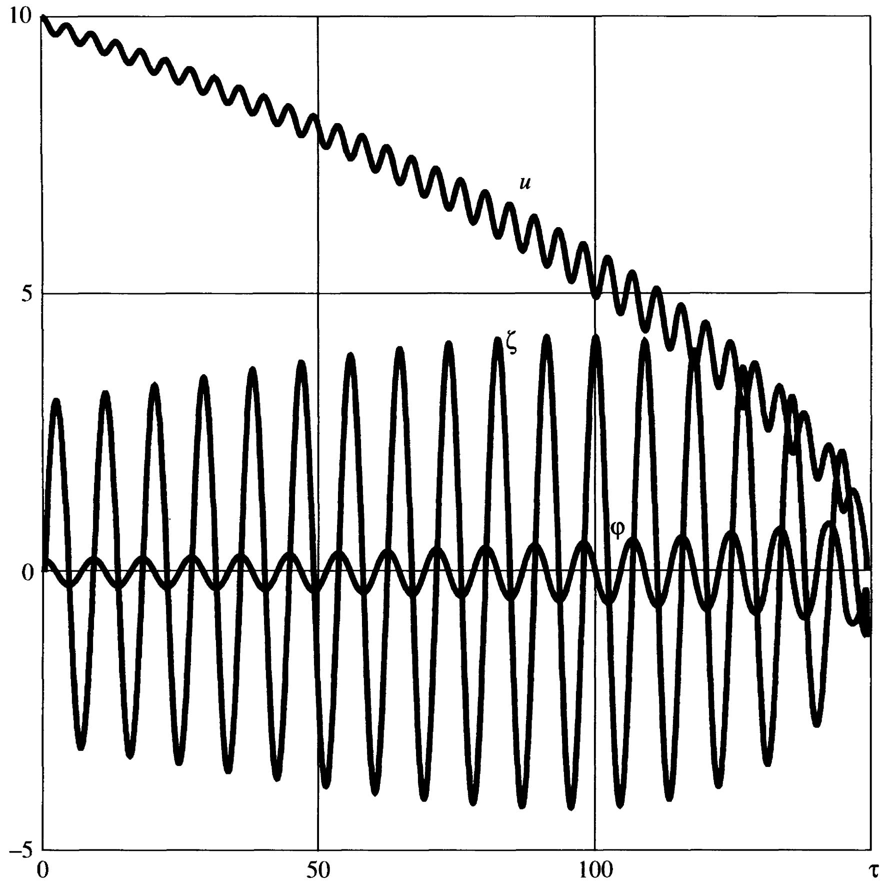

Fig. 5.

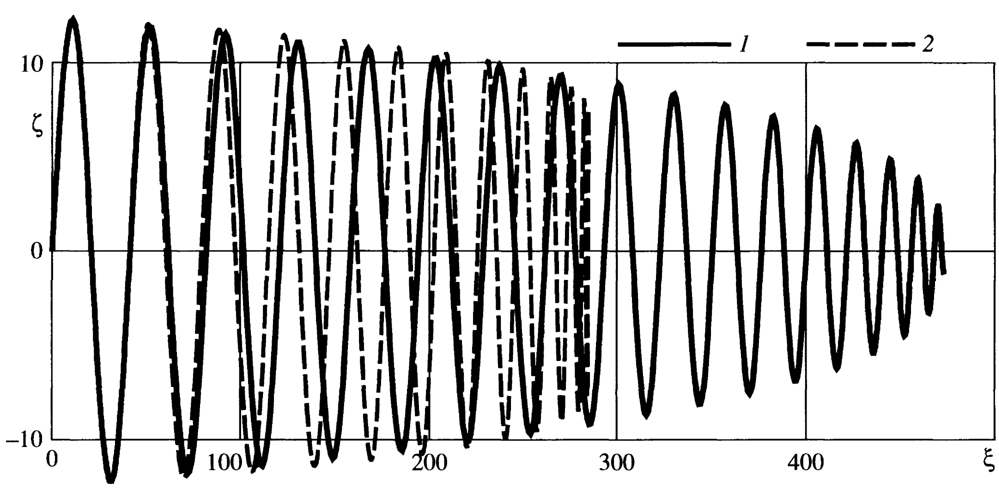

Fig. 6.

the body vanishes. In fact, the solution constructed ceases to be correct somewhat earlier, when $u \sim 1$, when the limits of applicability of the approximation employed, specified by condition (3.1), are violated.

We will now consider the motion of a body at a large angle to the horizon $\varphi = 1.078$ rad. In this case the transverse radiation force again reaches a maximum and is directed towards the x axis (see Fig. 2). The trajectories of the body motion in this case are shown in Fig. 6. Again, for comparison, we show the trajectories of the body both taking the action of the transverse component of the force into account (curve 1), and without it (curve 2), but taking into account the longitudinal component of the wave-drag force. In both cases the oscillations of the body attenuate monotonically, but now the body travels a longer path along the horizontal during the attenuation time when the transverse component of the radiation force is taken into account. This effect can also be explained using the formula for the function $G(\varphi)$: for large angles, this function decreases in modulus and also becomes positive, which corresponds to acceleration of the body along the horizontal axis. Note that, as follows from formula (2.5), the attenuation time of the motion T_0 is independent of the initial angle φ_0.

Oscillations of the angle of inclination of the velocity vector to the x axis, $\varphi(\tau)$, occurs with constant amplitude in the case considered until the motion is completed (compare with Fig. 5, where the amplitude of the oscillations $\varphi(\tau)$ grows). For large initial angles $\varphi_0 \in (1.078, \pi/2)$ the amplitude of these oscillations decreases with time, and when $\varphi_0 \in (0, 1.078)$ it grows.

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